On the Field Theoretical Approach to the Anomalous Scaling in Turbulence

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Anomalous scaling problem in the stochastic Navier-Stokes equation is treated in the framework of the field theoretical approach, successfully applied earlier to the Kraichnan rapid advection model. Two cases of the space dimensions $d = 2$ and $d \to \infty$, which allow essential simplification of the calculations, are analyzed. The presence of infinite set of the Galilean invariant composite operators with *negative* critical dimensions in the model discussed has been proved. It allows, as well as for the Kraichnan model, to justify the anomalous scaling of the structure functions. The explicit expression for the junior operator of this set, related to the square of energy dissipation operator, has been found in the first order of the ϵ expansion. Its critical dimension is strongly negative in the two dimensional case and vanishes while $d \to \infty$.

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It is well known that the Kolmogorov-Obuchov (KO) theory fails to describe certain features of fully developed turbulence. In particular, a good example is that of the single time structure functions of velocity

$$S_n(r) \equiv \langle [v_L(r) - v_L(0)]^n \rangle, \qquad v_L = \frac{(\boldsymbol{v}\boldsymbol{r})}{r}. \tag{1}$$

From the viewpoint of the KO theory, their behavior in the inertial range depends only on the mean energy dissipation rate $\bar{\varepsilon}$

$$S_n(r) = C_n(\bar{\varepsilon} r)^{n/3}, \tag{2}$$

which leads to linearity of the corresponding exponent in n. However, this is not consistent with experimental results. This fact, often referred to as anomalous scaling, is being actively discussed now. The problem is usually treated in the framework of phenomenological models [1], where it is related to large fluctuations of the energy dissipation rate. The deviations from the Kolmogorov scaling are described by additional (anomalous) exponents q_n as follows

$$S_n(r) \simeq C_n(\bar{\varepsilon} r)^{n/3}(r/L)^{q_n}, \tag{3}$$

where L is the integral scale of the flow. For $n > 3$ the experiment shows that q_n is notably less then zero.

The only statistical model allowing the calculation of anomalous exponents is the Kraichnan rapid advection model, introduced by Obukhov [2] and Kraichnan [3], which attracts considerable interest being a simple model for anomalous scaling investigation [4]. Recently it was treated within the field theoretical approach, based on the analysis of critical dimensions of so called composite operators — products of primary field and/or their derivative at a single point [5]. This approach provides one of the most efficient ways to determine the universal scaling quantities, by ignoring all inessential details, which is a great advantage in a complex problem investigation. Particularly, it became possible to calculate the anomalous exponents for the Kraichnan model in the second order of the ϵ expansion (the first order result in this case is consistent with that obtained in [4]). Another convenience of the approach in question is its universality: it does not exploit the peculiarity of the Kraichnan model and thus can be applied to a wide variety of the statistical models.

In the present letter this approach is developed for the stochastic Navier-Stokes (SNS) equation in two cases of space dimensions $d = 2$ and $d \to \infty$, which allow essential simplification of calculations. The obtained results are found to be analogous to that for the Kraichnan model. However, due to considerable technical complexity it is only qualitative analysis that can be performed.

The starting point in the scaling investigation is the dimensional analysis. We will distinguish momentum d^k and frequency d^ω dimensions, so the dimension of an arbitrary quantity F is represented as $[F] = (\text{length})^{-d^k_F} \times (\text{time})^{-d^\omega_F}$. It is obvious that $d^k_k = -d^k_r = 1$ and $d^\omega_\omega = -d^\omega_t = 1$. For the viscosity ν one obtains $d^k_\nu = -2$, $d^\omega_\nu = 1$. The fully developed turbulence has two characteristic lengths: the integral scale L, determined by the geometry of the flow, and the dissipative length $l \equiv \bar{\varepsilon}^{-1/4}\nu^{3/4}$. Thus, the natural representation for an arbitrary function $S(r)$ is

$$S(r) = \nu^{d^\omega_S} r^{-d_S} F\left(\frac{r}{l}, \frac{r}{L}\right). \tag{4}$$

Here $d_S \equiv d^k_S + 2d^\omega_S$ is so called canonical dimension of S, which plays an important role in the field theoretical approach (as well as conventional dimension for the static problems). Particularly, for the structure functions (1) $d^\omega_{S_n} = n$, $d^k_{S_n} = -n$ and consequently $d_{S_n} = n$, so the Eq. (4) takes the form

$$S_n(r) = \nu^n r^{-n} F_n\left(\frac{r}{l}, \frac{r}{L}\right). \tag{5}$$

However, this representation is not fruitful for the fully developed turbulence, because of the divergences of the functions F_n while the Reynolds number goes to infinity. More precisely, one is interested in the structure functions behavior in the inertial region $L \gg r \gg l$, which corresponds to the limit $r/l \to \infty$, $r/L \to 0$ in Eq. (5).

It should be noted that there are two different problems: the asymptotical behavior of the functions F_n while the first argument goes to infinity, and while the second one goes to zero.

The common way for treating the first problem is to isolate the factor $(r/l)^{-\gamma_n}$ in Eq. (5)

$$S_n(r) = \nu^n r^{-n} \left(\frac{r}{l}\right)^{-\gamma_n} \tilde{F}_n\left(\frac{r}{l}, \frac{r}{L}\right). \tag{6}$$

If the choice of the exponents γ_n can provide the condition $\tilde{F}_n(\infty, r/L) = \text{const} \neq 0$, then infrared scaling of structure functions $S_n(r)$ with *fixed* parameter r/L

$$S_n(r) = \lambda^{\Delta_n} S_n(r\lambda) \tag{7}$$

will be described by their critical dimensions $\Delta_n \equiv d_{S_n} + \gamma_n$.

In the framework of the KO theory, functions $\tilde{F}_n(r/L) \equiv \tilde{F}_n(\infty, r/L)$ are assumed to have nonzero limit while $r/L \to 0$ and the exponents γ_n are determined from the condition of the disappearing of ν dependence in the asymptotical behavior of the structure functions in the inertial region (this leads to the Eq. (2)).

On the other hand, critical dimensions of different quantities can be calculated directly from the corresponding stochastic equation by means of the renormalization group (RG) method. For the structure functions (1) the exact nonperturbative result can be obtained [6], which gives Kolmogorov's values $\Delta_n = -n/3$. However, the RG method does not require the finiteness of the functions $\tilde{F}_n(r/L)$ in Eq. (6) (the only condition is $l \ll L$). It is obvious that the anomalous scaling appears in the case of divergencies of the functions $\tilde{F}_n$ while $r/L \to 0$.

Within the field theoretical approach, the asymptotical behavior of the functions $\tilde{F}_n$ while $r/L \to 0$ can be analysed with the help of the operator product expansion (OPE), which allows to express the dependence discussed in terms of critical dimensions of composite operators. The detailed account for the RG and OPE methods applied to SNS equation can be found in [6], so here we provide only necessary information. In general, OPE of the discussed above functions $\tilde{F}_n$ has the form

$$\tilde{F}_n \propto \sum_{\Phi_k} C_n^{(k)} (r/L)^{\Delta_{\Phi_k}}, \tag{8}$$

where the coefficients $C_n^{(k)}$ are regular functions of r/L and thus can be treated as constants in the limit $r/L \to 0$. Summation in Eq. (8) is taken over all composite operators Φ_k, entering the OPE of S_n (in general, these are all possible operators allowed by the symmetry), and Δ_{Φ_k} are their critical dimensions. The latter, unlike the critical dimensions Δ_n of the structure functions, cannot be determined from the simple dimensionality considerations. In this case the application of the RG method is required, which allows to calculate nontrivial critical dimensions Δ_{Φ_k} as well as coefficients $C_n^{(k)}$ in the form of ϵ expansions.

One can see from Eq. (8) that the main contribution to F_n in the asymptotic region $r/L \to 0$ is given by terms with negative Δ_k, i. e. by composite operators with negative critical dimensions. These operators were called "dangerous" in [7], because the presence of such operators in the expansion (8) leads immediately to the divergences of the functions F_n while $r/L \to 0$ and, hence, to the anomalous scaling. As it was pointed out in [8], only Galilean invariant composite operators enter into the OPE of invariant objects, as the structure functions are. But it was not until now that invariant dangerous operators have been found in the SNS model. It is only the energy dissipation operator

$$\Phi_d = \frac{1}{2}\nu(\partial_i\varphi_k + \partial_k\varphi_i)^2, \qquad \boldsymbol{\varphi} = \boldsymbol{v} - \langle \boldsymbol{v} \rangle, \tag{9}$$

that was known to have zero critical dimension [6].

One of the main difficulties of the search of dangerous operators in the SNS model originates from the mixing of operators in renormalization. The matter is that an arbitrary ultraviolet finite renormalized operator is a linear combination of the primary one and those mixing in renormalization. This results in splitting of the whole set of operators into finite subsets — "families", closed under renormalization and containing operators of the same canonical dimensions and symmetry properties [6]. As a consequence, the renormalization of a composite operator Φ_i is characterized by a renormalization matrix Z_{ij}, rather than by a single constant as in the case of ordinary quantities

$$\Phi_i = \sum_j Z_{ij} \Phi_j^R. \tag{10}$$

Here the sum is taken over the full set of independent operators from the corresponding operator family. In fact, definite critical dimensions may be assigned to the so called basis operators only, which diagonalize the matrix Z_{ij}, and are determined by its eigenvectors.

Thus, the RG analysis of an operator family requires calculating of the full renormalization matrix from Eq. (10). This provides possibility to determine all basis operators and their critical dimensions. But the number of independent operators in the SNS model grows rapidly with the canonical dimension of the family, which makes the analysis of senior families extremely laborious.

Particularly, we will be interested in the scalar Galilean invariant operators as entering the OPE of the structure functions. Canonical dimensions of the operators in question are positive even integers. Up to the present moment the families with $d_\Phi = 2, 4, 6$ have been examined [8,9]. The mentioned above energy dissipation operator Φ_d belongs to the family with $d_\Phi = 4$, and has the minimal critical dimension $\Delta_{\Phi_d} = 0$ among the families discussed.

This fact encourages us to consider the families, containing powers of Φ_d, to be the most likely evidence of the presence of dangerous operators. The family with $d_\Phi = 8$, which contains the square of Φ_d, was examined in [10]. However, the analysis of the whole family was not performed due to technical difficulties (the family contains 12 independent operators). It was found out only that Φ_d^2 cannot be a basis operator because of its mixing with the others: the fact which breaks the equality

$$\Delta[\Phi_d^n] = n\Delta[\Phi_d], \tag{11}$$

proposed in [11].

We succeeded in the analysis of the discussed above family with canonical dimension 8 in two cases of space dimensions $d = 2$ and $d \to \infty$, where the essential simplification of the problem occurs. Let us consider the first case in more detail.

In this case the SNS equation for the velocity pulsation field $\boldsymbol{\varphi}$ can be reduced to the scalar equation

$$\partial_t \triangle\phi = \nu\triangle^2\phi - \partial_i\partial_m(\varepsilon_{nm}\partial_n\phi\partial_i\phi) + f \tag{12}$$

for the stream function ϕ defined by the relation $\varphi_i = \varepsilon_{ji}\partial_j\phi$ [12,13]. Here ϵ_{ij} is the second-rank antisymmetric tensor. Correlator of the random force f, which imitate the interaction of velocity pulsations with large-scale eddies, can be taken in the usual form

$$\langle f(x,t)f(x',t')\rangle = \delta(t-t') \times \int \frac{d\mathbf{k}}{(2\pi)^2}\, \exp\{i\mathbf{k}(\mathbf{x}-\mathbf{x}')\}\, D(\mathbf{k})\,, \tag{13a}$$

$$D(\mathbf{k}) = D_0 k^{4-2\epsilon}, \tag{13b}$$

where $\epsilon > 0$ is a formal small parameter of RG expansion. Its “physical” value $\epsilon = 2$ corresponds to the case of energy pumping. Correct RG approach for this model, developed in [12–14] gives the same results, as the standard SNS model.

Scalar formalism has at least two advantages: simpler calculation of each diagram and less number of independent operators. Let us see how the family of operators with canonical dimension $d_\Phi = 8$ can be treated within the scalar approach of the model (12), (13). To analyse an operator family one has to found the full set of independent operators. Operators should be constructed from the derivatives ∂_i, ∂_t of field ϕ convoluted with tensors ε_{ij}, δ_{ij} and must necessarily be true scalars (an essential demand as ε_{ij} is a pseudo-tensor and ϕ is a pseudo-scalar) and Galilean invariant. It can be shown that the consideration may be restricted to the case of operators irreducible to a full derivative of a junior one [6]. Also, the specific symmetries of two-dimensional space must be taken into account, leading to identities of the type $\varepsilon_{ij}\varepsilon_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$.

As the result one obtains 7 independent operators:

$$\begin{aligned}
&\Phi_1 = (\triangle\phi)^4,\\
&\Phi_2 = (\partial_i\partial_j\phi\,\partial_i\partial_j\phi)^2,\\
&\Phi_3 = \partial_i\partial_j\phi\,\partial_j\partial_k\phi\,\partial_k\partial_l\phi\,\partial_l\partial_i\phi,\\
&\Phi_4 = \varepsilon_{il}\,\triangle\partial_i\partial_j\phi\,\partial_j\partial_k\phi\,\partial_k\partial_l\phi,\\
&\Phi_5 = \varepsilon_{il}\,\nabla_t\partial_i\partial_j\phi\,\partial_j\partial_k\phi\,\partial_k\partial_l\phi,\\
&\Phi_6 = (\triangle^2\phi)^2,\\
&\Phi_7 = \nabla_t\partial_i\partial_j\phi\,\nabla_t\partial_i\partial_j\phi.
\end{aligned} \tag{14}$$

Here $\nabla_t \equiv \partial_t + \varepsilon_{ji}\partial_j\phi\partial_i$ is the substantial derivative. The square of energy dissipation operator is expressed in terms of operators (14) as follows

$$\Phi_d^2 = -\Phi_1 + 6\Phi_2 - 4\Phi_3. \tag{15}$$

We have calculated full 7×7 renormalization matrix Z_{ij} for the set of operators (14) in the first order of the ϵ expansion within the minimal subtraction scheme. As it was claimed in [10], Φ_d^2 is not a basis operator and does not have any certain critical dimension. But a dangerous operator still enters into the family in question. It is

$$\begin{aligned}\Phi'_d \cong -\Phi_1 + 6.03\Phi_2 - 4.04\Phi_3 - 0.58\Phi_4\\ + 0.06\Phi_5 + 0.10\Phi_6 - 0.02\Phi_7\end{aligned} \tag{16}$$

with critical dimension

$$\Delta_{\Phi'_d} \cong 8 - 5.92\epsilon. \tag{17}$$

At the “physical” point $\epsilon = 2$ it becomes $\Delta_{\Phi'_d} \cong -3.84$.

It is interesting to notice that the square of energy dissipation operator (15) makes the major contribution to the dangerous operator Φ'_d, while the other operators of the family (14) enter with small numerical coefficients. The same is the case with the critical dimension of Φ'_d: the major contribution $8 - 6\epsilon$ is that of diagonal element of the renormalization matrix, corresponding to the Φ_d^2, and the admixtion correction comprises only 0.08ϵ. So the operator (16) proves to be dangerous due to the square of the energy dissipation operator. However, it must be outlined that the discussed critical dimension $\Delta_{\Phi'_d}$ considerably deviates from $2\Delta_{\Phi_d} = 8 - 4\epsilon$ given by (11).

It is the second studied case of infinite dimensional turbulence, where the exclusive role of the energy dissipation operator becomes even more obvious. We have shown that this is the case when the renormalization matrix is block-triangular, so that the Φ_d^2 operator is the basis one. Its critical dimension

$$\Delta_{\Phi_d^2} = 8 - 4\epsilon \tag{18}$$

satisfies the Eq. (11) and equals to zero in the “physical” point as for the energy dissipation operator itself. Moreover, we show within one-loop approximation that in the case discussed the Eq. (11) proves to be valid for *all* powers of Φ_d [15].

Let us show now that the existence of one dangerous operator in the expansion (8) means that there is an infinite set of dangerous operators. Indeed, by combining inequality

$$|S_n|^{1/n} \leq |S_{n+1}|^{1/(n+1)}, \tag{19}$$

known from the probability theory with the definition of anomalous exponents (3), one obtains inequality for even exponents q_n

$$\frac{q_n}{n} \leq \frac{q_{n+2}}{n+2}, \qquad n = 2m. \tag{20}$$

Since q_n are negative (which follows from both the experimental data and the presence of a dangerous operator in the OPE of structure functions) the inequality (20) implies that the absolute value of q_n grows more rapidly than n. Taking into account the expansion (8) we see that only infinite set of dangerous operators meet the last condition.

Thus, the results, obtained for the SNS model, proved to be quite similar to those for the Kraichnan model [5]. In both cases the anomalous scaling of the structure functions can be considered as a consequence of the infinite set of dangerous operators in the corresponding OPE, related to the powers of the energy dissipation operator. Within the used approach, both models exhibit the particularity of the infinite-dimensional case, were the critical dimensions of all known dangerous operators vanish (at least within the considered approximations). In the Kraichnan model this is in agreement with the disappearing of anomalous scaling and intermittency while $d \to \infty$ [16]. Our results make us expect that the analogous behavior takes place in the SNS model also, which was first discussed in [17], where the possible use of $1/d$ expansion for real turbulence was pointed out.

However, the Kraichnan model contains at least two principal simplifications. Firstly it is free of problems connected with the mixing of operators: the critical dimension of any power of the dissipation operator is determined by the corresponding diagonal element of the renormalization matrix (due to its triangularity). This allows to calculate the critical dimensions of all operators discussed in ϵ expansion [5], which seems impossible for the SNS model. Secondly, the expansion (8) for any certain structure function S_n is finite in the Kraichnan model, which eliminates the necessity of its summation. The anomalous exponent q_n in this case simply equals to the minimal Δ_k in Eq. (8). However there is no reason to expect this in the SNS model, so the additional problem of the summation of the expansion (8) arises. As a consequence, the asymptotical behavior of the structure functions can be more complicated than powerlike.

It becomes clear now that the developed here approach gives qualitative account for anomalous scaling in the SNS model, though, due to extreme difficulty of the problem it requires some additional ideas for further investigations. In this context the simple result for the infinite-dimensional case is of particular interest. It implies the actuality of the detailed study of this case as the starting point of $1/d$ expansion.

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